Hw4 Math 2230 B/C

P170.1. a) since $\frac{\pi}{2}i$ is inside C, e^{-z} is analytic inside and on C

Apply Cauchy Integral formula to get:

$$\int_C \frac{e^{-z}}{z-\frac{\pi}{2}i}\,dz = 2\pi i\,(e^{-z})\Big|_{z=\frac{\pi}{2}i} = 2\pi i(-i) = 2\pi$$

b) $z^2+8=0 \iff z=\pm 2\sqrt{2}i$ outside C

$\therefore \frac{\cos z}{z^2+8}$ is analytic inside and on C

$$\therefore \int_C \frac{\cos z/z^2+8}{z}\,dz = 2\pi i\left(\frac{\cos z}{z^2+8}\right)\Big|_{z=0} = \frac{\pi}{4}i$$

c) $$\int_C \frac{z/2}{z+\frac{1}{2}}\,dz = 2\pi i\left(\frac{z}{2}\right)\Big|_{z=-\frac{1}{2}} = -\frac{\pi}{2}i$$

d) $$\int_C \frac{\cosh z}{z^4}\,dz = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\left(\frac{e^z+e^{-z}}{2}\right)\Big|_{z=0}$$

$$= \frac{\pi}{3}i\left(\frac{e^z-e^{-z}}{2}\right)\Big|_{z=0}$$

$= 0$

e) $\tan\left(\frac{z}{2}\right) = \frac{\sin(z/2)}{\cos(z/2)}$ $\quad \cos(z/2)=0 \iff z=(1+2n)\pi$ is outside C

$\therefore \tan\left(\frac{z}{2}\right)$ is analytic in and on C

$$\therefore \int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = 2\pi i\cdot\frac{d}{dz}\left(\tan\frac{z}{2}\right)\Big|_{z=x_0}$$

$$= 2\pi i\left(\frac{1}{2}\sec^2\frac{z}{2}\right)\Big|_{z=x_0} = \pi i\sec^2\frac{x_0}{2}$$

3. since 2 is inside C

$$\int_C \frac{2s^2-s-2}{s-2}\,ds = 2\pi i\,(2s^2-s-2)\Big|_{s=2} = 8\pi i$$

If $|z|>3$, then $\frac{2s^2-s-2}{s-z}$ is analytic $\Rightarrow g(z)=0$

7. $\int_C \frac{e^{az}}{z}\,dz = 2\pi i\,(e^{az})\Big|_{z=0} = 2\pi i$

$$\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} \frac{e^{az(\theta)}}{z(\theta)} z'(\theta)\,d\theta = \int_{-\pi}^{\pi} e^{a\cos\theta + ia\sin\theta} \cdot i\,d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}\left(\cos(a\sin\theta) + i\sin(a\sin\theta)\right)d\theta$$

$$= -\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\,d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta$$

By comparing Real / Imaginary part

we get $\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\pi$.

10. $\forall z_0 \in \mathbb{C}$, let C_R be a circle centered at z_0 with with Radius R

$$|f''(z_0)| \le \frac{2!\cdot \max_{z\in C_R}|f(z)|}{R^2} \le \frac{2\cdot \max_{z\in C_R} A|z|}{R^2} \le \frac{2\cdot A(|z_0|+R)}{R^2} \to 0 \quad \text{as } R\to\infty$$

$\therefore f''(z_0)=0 \quad \forall z_0 \in \mathbb{C}$

$\therefore f(z) = a_1 z + a_2$ and $|f(z)| \le A|z| \Rightarrow a_2 = 0$.

P195. 1. $\cosh(z) = \dfrac{e^z + e^{-z}}{2}$

$$\cosh(z^2) = \frac{e^{z^2} + e^{-z^2}}{2} = \frac{\sum_{n=0}^{\infty} \frac{z^{2n}}{n!} + \sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!}}{2} = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!}$$

$$\therefore z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

2. a) f is analytic on disk $|z-1| < \infty$. $f(z) = e^z$, $f^{(n)}(z) = e^z$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = \sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

b) $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$, $|w| < \infty$ $\Rightarrow e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \Rightarrow e^z = e \cdot \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$

P219. 3. $\dfrac{1}{z} = \dfrac{1}{2+(z-2)} = \dfrac{1}{2} \cdot \dfrac{1}{1+(z-2)/2} = \dfrac{1}{2} \cdot \sum_{n=0}^{\infty} \left(-\dfrac{z-2}{2}\right)^n$

$$\Rightarrow \frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2} = -\frac{1}{4}\sum_{n=1}^{\infty} n \cdot \left(-\frac{z-2}{2}\right)^{n-1}$$

$$\therefore \frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty} (n+1) \cdot (-1)^n \cdot \left(\frac{z-2}{2}\right)^n.$$

4. $f(z) = (1-\cos z)/z^2 = \dfrac{1 - \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}}{z^2} = -\sum_{n=1}^{\infty} (-1)^n \dfrac{z^{2n-2}}{(2n)!}$

$$= -\sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2n}}{(2n+2)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+2)!}$$

So if we define $f(0) = \frac{1}{2}$, which is just the function in the question. It becomes a Tylor Expansion on $\mathbb{C}$, and so analytic

P177. 2. let $g(z) = \frac{1}{f(z)}$ analytic on R. $g(z)$ is non-constant

$\Rightarrow$ maximum of $|g(z)|$ is attained somewhere on the boundary

$\Rightarrow$ minimum of $|f(z)| = \frac{1}{|g(z)|}$ is attained somewhere on the boundary

4. For $0 \le x \le \pi$, $\sin^2 x$ attains maximum when $x = \frac{\pi}{2}$

while $\sinh y = \frac{e^y - e^{-y}}{2}$ is non-negative and strictly increasing for $0 \le y \le 1$

$\therefore |f(z)|^2 = \sin^2 x + \sinh^2 y$ attains its maximum when $z = \frac{\pi}{2} + i$

5. let $g(z) = e^{f(z)} = e^{u(x,y)} \cdot e^{iv(x,y)}$ is analytic and non-zero on R

$\therefore$ By Question (2) $|g(z)| = |e^{u(x,y)}| = e^{u(x,y)}$ has a minimum on the boundary of R.

Therefore, $u(x,y)$ attains its minimum on the boundary.

P186. 9. a) $|z_n| = |z + (z_n - z)| \le |z| + |z_n - z|$

Take N s.t. $\forall i \ge N$ $d(z_i, z) \le \frac{1}{2}$. or $z_i \in$ ~~$B(z, \frac{1}{2})$~~ $B_{\frac{1}{2}}(z)$, let $M = |z| + \frac{1}{2}$

Then $|z_N| \le |z| + \frac{1}{2} = M$.

b) since $z_n \to z$, $x_n \to x$, $y_n \to y$. $\Rightarrow$ $|x_n| \le M_1$ and $|y_n| \le M_2$

$\Rightarrow x_n^2 \le M_1^2$ $y_n^2 \le M_2^2$ $\Rightarrow$ $|z_n|^2 = |x_n^2 + y_n^2| \le M_1^2 + M_2^2$.